ISSN (e): 2250 – 3005 || Volume, 06 || Issue, 7|| July – 2016 ||
International Journal of Computational Engineering Research (IJCER)

Smooth Compositions with a Nonsmooth Inner Function

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ABSTRACT. Let $p : \mathbb{R} \to \mathbb{R}$ be a given function, and let A_p be the set of smooth functions f such that $f(p(\cdot) + c)$ is smooth for any $c \in \mathbb{R}$. We show that if p is not smooth, then either every element of A_p is constant, or there is a nonzero constant d such that A_p equals to the set of smooth functions of periodicity d.

1. INTRODUCTION

In this paper, we prove the following result:

Theorem 1.1. *Let n be a nonnegative integer or ∞, $p : \mathbb{R} \to \mathbb{R}$ be a given function with $p \notin C^n(\mathbb{R}, \mathbb{R})$ and*

(1.1) $A_p = \{f \in C^n(\mathbb{R}, \mathbb{R}) \mid f(p(\cdot) + c) \in C^n(\mathbb{R}, \mathbb{R})$ *for any* $c \in \mathbb{R}\}$.

Then, either $A_p = \mathbb{R}$ or $A_p = C^n_d(\mathbb{R}, \mathbb{R})$ for some nonzero constant d. Here, $A_p = \mathbb{R}$ means that every element in A_p is constant, and $C^n_d(\mathbb{R}, \mathbb{R})$ means the collection of all functions in $C^n(\mathbb{R}, \mathbb{R})$ of periodicity d.

This result is motivated by the work [1] of Christensen and Wu on diffeological vector spaces. A weaker form of Theorem 1.1 is needed in [1]. The method in [1] dealing with the weaker form seems not applicable to prove Theorem 1.1.

It is clear that A_p is a translation invariant subalgebra of $C^n(\mathbb{R}, \mathbb{R})$. As an example, take $p = \chi_E$ with E an abitrary subset of $\mathbb{R}$ ($E \neq \emptyset, \mathbb{R}$). Then, it is clear that $\cos(2\pi x), \sin(2\pi x) \in A_p$. In fact, it is not hard to see that $A_p = C^n_1(\mathbb{R}, \mathbb{R})$ in this specific example.

A key ingredient in the proof of Theorem 1.1 is the following result about the continuity of maps with σ-compact graph.

Lemma 1.1. *Let X be a Hausdorff Baire space, Y be a topological space and $f : X \to Y$ be a map with σ-compact graph. Then, f is continuous on a dense open subset of X.*

For a Baire space, we mean a topological space satisfying Baire's category theorem. That is, a countable intersection of open dense subset is still dense. Typical examples of Baire spaces are complete metric spaces and locally compact Hausdorff spaces.

Acknowledgement. The authors would like to thank the referee for generously sharing ideas that significantly strengthen the results (both Theorem 1.1 and Lemma 1.1) of the previous version of this paper, and to thank their colleague, Prof. Enxin Wu, for many helpful discussions.

2. Proof of Theorem 1.1

We first prove Lemma 1.1.

Proof of Lemma 1.1. Let

$$\Gamma = \{(x, f(x)) \in X \times Y \mid x \in X\} \tag{2.1}$$

be the graph of f. Suppose that $\Gamma = \cup_{n=1}^{\infty} K_n$ where K_n is a compact subset of $X \times Y$ for $n = 1, 2, \cdots$. Then, $A_n = \pi_X(K_n)$ is a compact subset of X, where $\pi_X : X \times Y \to X$ is the natural projection. Note that $f|_{A_n} : A_n \to Y$ as a map with compact graph is continuous. To see this, let F be any closed subset of Y, then

$$(f|_{A_n})^{-1}(F) = \pi_X(K_n \cap (X \times F)) \tag{2.2}$$

is compact and hence closed.

Let $F_n = A_n \setminus \text{Int}(A_n)$ where $\text{Int}(A_n)$ is the interior of A_n. Then, $U_n = X \setminus F_n$ is a dense open subset of X. So, $\cap_{n=1}^{\infty} U_n$ is a dense subset of X by that X is a Baire space. On the other hand, since $X = \cup_{n=1}^{\infty} A_n$,

$$\cap_{n=1}^{\infty} U_n \subset \cup_{n=1}^{\infty} \text{Int}(A_n). \tag{2.3}$$

Hence $\cup_{n=1}^{\infty}\text{Int}(A_n)$ is a dense open subset of X. Moreover, f is continuous at the points in $\cup_{n=1}^{\infty}\text{Int}(A_n)$ since $f|_{A_n}$ is continuous for $n = 1, 2, \cdots$. This completes the proof of the Lemma. Q

For clarity, we will separate the proof of Theorem 1.1 into two cases:(i) graph of p is closed and (ii) graph of p is not closed.

Lemma 2.1. *Let notations be the same as in Theorem 1.1. Suppose that the graph of p is closed, then $A_p = \mathbb{R}$.*

Proof. We will proceed by contradiction. Assume that $A_p \neq \mathbb{R}$. Let $G \subset \mathbb{R}$ be the largest open subset of $\mathbb{R}$ such that p is continuous on G. By Lemma 1.1, G is dense in $\mathbb{R}$. Let $F = \mathbb{R} \setminus G$. For clarity, we divide the proof into several claims.

Claim 1. *By the structure of open subsets in $\mathbb{R}$, G is a disjoint union of countably many open intervals. Let (a, b) be one of such open intervals where a may be $-\infty$ and b may be $+\infty$. Then, $p \in C((a, b))$.*

Proof of Claim 1. We only show the case that $a = -\infty$ and b is finite. The proofs of the other cases are similar. In this case, $(a, b) = (-\infty, b]$.

Because the graph of p is closed and $p \in C((-\infty, b))$, the asymptotic behavior of p as x approaching b^- happens in only three cases:

(1) $\lim_{x\to b^-} p(x) = p(b)$,
(2) $\lim_{x\to b^-} p(x) = +\infty$, and
(3) $\lim_{x\to b^-} p(x) = -\infty$.

To prove Claim 1, we only need to exclude (2) and (3). If (2) is true, we have

$$\lim_{y\to+\infty} f(y) = \lim_{x\to b^-} f(p(x) + a) = f(p(b) + a) \tag{2.4}$$

for any $a \in \mathbb{R}$ and $f \in \mathcal{A}_p$. This means that $\mathcal{A} = \mathbb{R}$ which is a contradiction. Case (3) can be excluded similarly.

Claim 2. *F has no isolated points.*

Proof of Claim 2. Suppose that x_0 is an isolated point of F. Then, by Claim 1, p is continuous on a neighborhood of x_0. So, $x_0 \in G$ which is a contradiction.

Claim 3. *Any point of continuity for $p|_F : F \to \mathbb{R}$ is a point of continuity for p.*

Proof of Claim 3. Let $x_0 \in F$ be a point of continuity for $p|_F$. If x_0 is not a point of continuity for p. Then there is a $\epsilon_0 > 0$ and a sequence $\{x_1, x_2, \cdots, x_n, \cdots\}$ of points tending to x_0 as $n \to \infty$, such that

$$|p(x_n) - p(x_0)| \geq \epsilon_0 \tag{2.5}$$

for any $n = 1, 2, \cdots$. Since x_0 is a point of continuity for $p|_F$, x_n F for n large enough. Moreover, since x_0 is not isolated in F (by Claim 2) and by Claim 1, $x_n \in (a_n, b_n)$ where (a_n, b_n) is an open interval in the disjoint open interval decomposition of G for n large enough. It is clear that $a_n \to x_0$ and $b_n \to x_0$ as $n \to \infty$. Furthermore, since x_0 is a point of continuity for $p|_F$ and $a_n \in F$,

$$|p(a_n) - p(x_0)| < \epsilon_0/2 \tag{2.6}$$

for n large enough. Since $p \in C([a_n, b_n])$ (by Claim 1), and by (2.5) and (2.6), there is a point $\tilde{x}_n \in [a_n, b_n]$ such that

$$p(\tilde{x}_n) = p(x_0) + \epsilon_0/2 \tag{2.7}$$

or

(2.8) $$p(\tilde{x}_n) = p(x_0) - \epsilon_0/2.$$

Then there is a sequence $\{\tilde{x}_1, \tilde{x}_2, \ldots\}$ of points tending to x_0 as $n \to \infty$ (taking subsequence if necessary) such that

(2.9) $$p(\tilde{x}_n) \to p(x_0) + \epsilon_0/2$$

or

(2.10) $$p(\tilde{x}_n) \to p(x_0) - \epsilon_0/2$$

as $n \to \infty$. This contradicts the fact that the graph of p is closed

Claim 4. *p is continuous on* $\mathbb{R}$.

Proof of Claim 4. Assume that $F \neq \emptyset$ By Lemma 1.1, $p|_F : F \to \mathbb{R}$ is continuous on some dense open subset of F . So, there is an open interval I such that $p|_F$ is continuous on $I \cap F$ with $I \cap F \neq \emptyset$. By Claim 3, points in $I \cap F$ are points of continuity for p. So p is continuous on I which implies that $I \subset G$ and contradicts $I \cap F \neq \emptyset$.

We are now ready to complete the proof. Since $p \notin C^n(\mathbb{R}, \mathbb{R})$, we know that $n \geq 1$. Let $f \in \mathrm{A}_p$ be a nonconstant function. Then, $f'(y_0) = 0$ for some $y_0 \in \mathbb{R}$. It is clear that

(2.11) $$\tilde{f}(y) = f(y - p(x_0) + y_0)$$

also belongs to A_p and

(2.12) $$\tilde{f}'(p(x_0)) = f'(y_0) \neq 0.$$

Let $g(x) = \tilde{f}(p(x))$. Then $g \in C^n(\mathbb{R}, \mathbb{R})$ because $\tilde{f} \in \mathrm{A}$. Since p is continuous at x_0,

(2.13) $$p(x) = \tilde{f}^{-1}(g(x))$$

for every x in some neighborhood of x_0. This implies that p is C^n in some neighborhood of x_0. Because x_0 was arbitrary, $p \in C^n(\mathbb{R}, \mathbb{R})$ which is a contradiction. Q

Lemma 2.2. *Let the notation be the same as in Theorem 1.1. If the graph of p is not closed, then either* $\mathrm{A}_p = \mathbb{R}$ *or* $\mathrm{A}_p = C^n_d(\mathbb{R}, \mathbb{R})$.

Proof. Let

(2.14) $L = \{d \in \mathbb{R} \mid f(y + d) = f(y)$ for any $y \in \mathbb{R}$ and any $f \in \mathrm{A}_p\}$.

It is clear that L is closed subgroup of $\mathbb{R}$ since $\mathrm{A}_p \subset C(\mathbb{R}, \mathbb{R})$. Let

(2.15) $\Lambda(x) = \{y \in \mathbb{R} \mid y = \lim_{n\to\infty} p(x_n)$ for some sequence $x_n \to x\}$.

Claim 1. *For any $y \in \Lambda(x)$, $y - p(x) \in L$.*
Proof of Claim 1. Let $\{x_n\}$ be a sequence of points tending to x such that

$$(2.16) \qquad p(x_n) \to y.$$

Then, for any $f \in A_p$,

$$(2.17) \qquad f(p(x) + c) = \lim_{n\to\infty} f(p(x_n) + c) = f(y + c)$$

for any $c \in \mathbb{R}$ because $f(p(\cdot) + c)$ is continuous. This means that $y - f(x) \in L$.

Because the graph of p is not closed, there is a point $x \in \mathbb{R}$, such that $\Lambda(x) \ni \{p(x)\}$. Then, by Claim 1, we know that $L \quad \{0\}$. Therefore, $L = \mathbb{R}$ or $L = d\mathbb{Z}$ for some nonzero constant d. For the first case, we know that $A_p = \mathbb{R}$. For the second case, we have $A_p \subset C^n_d(\mathbb{R}, \mathbb{R})$.

Let $\pi : \mathbb{R} \to \mathbb{R}/d\mathbb{Z}$ be the natural projection. It is clear that for any $f \in C^n_d(\mathbb{R}, \mathbb{R})$, f descends to a function $f : \mathbb{R}/d\mathbb{Z} \to \mathbb{R}$ such that

$$(2.18) \qquad f = \overline{f} \circ \pi.$$

Moreover, $\overline{f} \in C^n(\mathbb{R}/d\mathbb{Z}, \mathbb{R})$.
Claim 2. *For any constant c, $\pi(p(\cdot) + c) : \mathbb{R} \to \mathbb{R}/d\mathbb{Z}$ is continuous.*
Proof of Claim 2. Let $x_n \to x$, we want to show that

$$\pi(p(x_n) + c) \to \pi(p(x) + c).$$

Suppose this is not true. Then, by the compactness of $\mathbb{R}/d\mathbb{Z}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\pi(p(x_{n_k}) + c) \to y \in \mathbb{R}/d\mathbb{Z},$$

as $n \to \infty$, where $y \neq \pi(p(x) + c)$. Let $z \in \mathbb{R}$ be such that $\pi(z) = y$. Then

$$z - p(x) - c \notin d\mathbb{Z}.$$

On the other hand, for any $f \in A_p \subset C^n_d(\mathbb{R}, \mathbb{R})$,

$$(2.19) \qquad f(p(x_{n_k}) + c) = f \circ \pi(p(x_{n_k}) + c) \to \overline{f}(y) = f(z).$$

Moreover, since $f(p(\cdot) + c)$ is continuous, we have

$$(2.20) \qquad f(p(x_{n_k}) + c) \to f(p(x) + c),$$

and thus $f(z) = f(p(x)+ c)$. Since this is true for every $f \in A_p$ and A_p is translation invariant, every element of A_p of periodicity $z - p(x) - c$. That is, $z - p(x) - c \in d\mathbb{Z}$, which is a contradiction.

We are not ready to complete the proof.

When $n = 0$, for any $f \in C^0_d(\mathbb{R}, \mathbb{R})$, we know that

$$(2.21) \qquad f(p + c) = \overline{f} \circ \pi(p + c) \in C^0(\mathbb{R}, \mathbb{R}).$$

This means that $f \in A_p$. So, we have shown that $A_p = C^n_d(\mathbb{R}, \mathbb{R})$ for the case $n = 0$.

When $n \geq 1$, if $A_p \quad \mathbb{R}$, let f_0 be a nonconstant function in A_p. Then a similar argument as in the proof of Lemma 2.1 (after Claim 4 in the proof of Lemma 2.1) using inverse function theorem will show that $\pi(p + c) \in C^n(\mathbb{R}/d\mathbb{Z}, \mathbb{R})$. Then, for any $f \in C^n_d(\mathbb{R}, \mathbb{R})$,

$$(2.22) \qquad f = \overline{f} \circ \pi(p + c) \in C^n(\mathbb{R}, \mathbb{R}).$$

This means that $f \in A_p$. So $A_p = C^n_d(\mathbb{R}, \mathbb{R})$. Q

Proof of Theorem 1.1. The combination of Lemma 2.1 and Lemma 2.2 gives us Theorem 1.1. Q

References

[1] Christensen, J. D.; Wu, Enxin. *Diffeological vector spaces*. arXiv:1703.07564.